Final Spin from Binary Black Hole Coalescence

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Numerical relativity

- We are interested to accurately determine the gravitational wave content and physical properties of spacetimes which are:
  - Strong.
  - Dynamical.
  - Without symmetries.

- In the strong-field, dynamical regime, nonlinear terms of the Einstein equation play an important role – approximations break down.

- We resort to numerical computation (computer simulation) to determine solutions of the Einstein equations:

\[ R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R = 8\pi T_{\alpha \beta} \]

- For the purpose of this talk, we consider only vacuum solutions, i.e. \( T_{\alpha \beta} = 0 \).
BSSN 1st order in space, 2nd order in time
1+log lapse
For the Bona-Massó slicing conditions:
\[(\partial_t - \beta^i \partial_i)\alpha = -\alpha^2 f(\alpha) K\]

we set \(f = 2/\alpha\)

\(\Gamma\)-driver shift evolution
\[\partial_t \beta^i = k \partial_t \tilde{\Gamma}^i \quad (k > 0)\]

Carpet Adaptive Mesh Refinement follows Puncture movement
Wave Extraction both with Zerilli Extraction and the Newman-Penrose \(\Psi_4\)
Puncture initial data with PN derived orbital parameters
Assume a conformal factor of the form:
\[\psi = u + \sum_{i=1}^{N} \frac{m_i}{2r_i}.\]

Find \(C^2\) solutions for \(u\) of the Hamiltonian constraint:
\[\tilde{\nabla}^2 u + \frac{1}{8} \chi^7 \tilde{A}_{ij} \tilde{A}^{ij} (1 + \chi u)^{-7} = 0\]
BH Astrophysics with Numerical Relativity

- There are a number of interesting physics results available from studying the last orbits, plunge and ringdown.
  - State of the final BH from generic initial data
  - Recoil of the final BH
  - Mode decomposition of the plunge waveform.
- These results are easily accessible, given reasonable quasi-circular/PN orbit parameters at late times.

In a series of papers, we have studied the merger physics of binary-BHs with spins:
- Rezzolla et al., “The final spin from the coalescence of aligned-spin black-hole binaries”,
- Rezzolla et al., “On the final spin from the coalescence of two black holes”,

Jennifer Seiler jese@aei.mpg.de  BBH Final Spin Formula
The objective of this talk is to derive a phenomenological formula for spin of a black hole resulting from the merger of two black holes of arbitrarily oriented spins and generic mass ratio. This has applications for:

- statistical distribution of black hole properties
- simulations of the central regions of galaxies
- dynamics of star clusters

We need to simulate 2 spinning black holes over a 7D parameter space

\[ \{S_1^i, S_2^i, M_1/M_2\} \]

to get one final black hole

\[ \{v_{kick}^i\} \{S_{fin}^i/M_{fin}^2\} \]
Black Hole Spins

- We obtain a general 2nd order polynomial expansion with 5 restricting assumptions for our coefficients:
  - mass radiated in gravitational waves may be neglected, \( M_{\text{fin}} \approx M \):
    \[
    \frac{M_{\text{rad}}}{M} = 1 - \frac{M_{\text{fin}}}{M} \approx 5 - 7 \times 10^{-2}
    \]
  - magnitude of the final spin vector is the sum of the initial spin vectors plus a third vector, \( \vec{\ell} \):
    \[
    S_{\text{fin}} = S_1 + S_2 + \vec{\ell}
    \]
    \[
    \ell = L - J_{\text{rad}}
    \]
    - The vector \( \vec{\ell} \) is parallel to \( L \) with a resulting error in the estimate of \( \sim |J_{\text{rad}}^\perp|^2 / |\vec{\ell}|^2 \sim |J_{\text{rad}}^\perp|^2 / (2\sqrt{3}M_1M_2)^2 \) these errors are small in all the configurations that we have analysed
    - When the initial spin vectors are equal and opposite \( (S_1 = -S_2) \) and the masses are equal \( (q = 1) \), the spin of the final black hole is the same as for the non-spinning binaries
    - The extreme mass ratio limit (EMRL) is trivial
      \[
      S_{\text{fin}} = S_1 \quad \text{if} \quad M \to 0
      \]
Black Hole Spins

- Using these assumptions, it follows that:

\[
|a_{\text{fin}}| = \frac{1}{(1 + q)^2} \left[ |a_1|^2 + |a_2|^2 q^4 + 2|a_2||a_1|q^2 \cos \alpha + 2 \left( |a_1| \cos \beta + |a_2|q^2 \cos \gamma \right) |\ell|q + |\ell|^2 q^2 \right]^{1/2},
\]

where \( \cos \alpha \equiv \hat{a}_1 \cdot \hat{a}_2 \), \( \cos \beta \equiv \hat{a}_1 \cdot \hat{\ell} \), \( \cos \gamma \equiv \hat{a}_2 \cdot \hat{\ell} \).

- In order to obtain \( |\ell| \) we need to match this equation against general second order polynomial expansions for:
  - Equal mass, unequal but aligned spin binaries
  - Unequal mass, equal spin binaries

\[
|\ell| = \frac{s_4}{(1 + q^2)^2} \left( |a_1|^2 + |a_2|^2 q^4 + 2|a_1||a_2|q^2 \cos \alpha \right) + \left( \frac{s_5 \nu + t_0 + 2}{1 + q^2} \right) \left( |a_1| \cos \beta + |a_2|q^2 \cos \gamma \right) + 2\sqrt{3} + t_2 \nu + t_3 \nu^2.
\]

- Numerical simulations to obtain \( s_4, s_5, t_0, t_2, t_3 \).
- Test against generic misaligned spin binaries.
Final spin via horizon shape

- Valid once a common horizon has formed and settled down to a perturbed state.
- Measure equatorial circumference $C_e$ and polar circumference $C_p$ along orthogonal great circles.
- $C_r = C_p/C_e$ settles to a constant value:

$$C_r(j) = \frac{1 + \sqrt{1 - j^2}}{\pi} E \left( - \frac{j^2}{(1 + \sqrt{1 - j^2})^2} \right)$$

where $j = a/M$, and $E(k)$ is the complete elliptic integral of the second kind

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k \sin^2 \theta} \, d\theta.$$  

- This equation is integrated numerically to obtain $j$ from the horizon shape.
Parameter studies with spinning black holes

Aligned spin leads to an orbital hang-up.

<table>
<thead>
<tr>
<th>r0</th>
<th>r1</th>
<th>r2</th>
<th>r3</th>
<th>r4</th>
<th>r5</th>
<th>r6</th>
<th>r7</th>
<th>r8</th>
</tr>
</thead>
</table>

![Graphs showing spiral orbits for different spin alignments with corresponding time series plots for $h_+$](image)

- $x(M)$ and $y(M)$ axes for each graph
- $r_0$, $r_4$, $r_8$ labels
- Time series $h_+$ plots for different spins
Equal Mass, Aligned Unequal Spin Binaries

- We have carried out studies in the parameter space of equal-mass aligned spin binaries, starting from non-eccentric orbit.
- Vary the spin of each BH from $a = -0.6$ to $a = +0.6$.
- Initial studies determined final BH parameters (final spin, radiated energy, kick) as a function of binary parameters.
- Kick depends quadratically on the spin difference, up to $\sim 450\, km/s$ in the maximal case.
- Final spin is an almost linear function of the initial spins.

Spin of the final BH.
Aligned Unequal Spins, Equal Mass

The resulting expression is:

\[ a_{\text{fin}} = p_0 + p_1(a_1 + a_2) + p_2(a_1 + a_2)^2. \]

with

\[ p_0 = 0.6883 \pm 0.0003, \quad p_1 = 0.1530 \pm 0.0004, \quad p_2 = -0.0088 \pm 0.0005, \]

\[ p_0 = \frac{\sqrt{3}}{2} + \frac{t_2}{16} + \frac{t_3}{64}, \quad p_1 = \frac{1}{2} + \frac{s_5}{32} + \frac{t_0}{8}, \quad p_2 = \frac{s_4}{16}. \]

predicts a minimum and maximum spin:

\[ (a_{\text{fin}})_{\text{min}} \approx 0.347 \]
\[ (a_{\text{fin}})_{\text{max}} \approx 0.959 \]

for aligned spins.
Unequal Mass, Aligned Spins

- The spin of the final black hole has been determined for very generic initial conditions:
  - Arbitrary aligned spins
  - Unequal masses
- In the extreme-mass-ratio limit, approximation methods can be used.

\[ a_{\text{fin}} = a + s_4 a^2 \nu + s_5 a \nu^2 + t_0 a \nu + 2 \sqrt{3} \nu + t_2 \nu^2 + t_3 \nu^3 \]

\[ s_4 = -0.129 \pm 0.012 \]
\[ s_5 = -0.384 \pm 0.261 \]
\[ t_0 = -2.686 \pm 0.065 \]
\[ t_2 = -3.454 \pm 0.132 \]
\[ t_3 = 2.353 \pm 0.548 \]
Accuracy for Aligned spins

- Numerical relativity results for non-spinning BHs (Jena, Goddard, Penn State)
- Extreme mass ratio calculations for the $m_1 \gg m_2$ limit (Buonanno-Kidder-Lehner 2007)

$$a_{\text{fin}} = a + s_4 a^2 \nu + s_5 a \nu^2 + t_0 a \nu + 2\sqrt{3}\nu + t_2 \nu^2 + t_3 \nu^3$$

**Equal-mass, spinning**

**Unequal mass, non-spinning**

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BBH Final Spin Formula
Accuracy for Misaligned Spins

Aligned spinning

- Circles refer to equal-mass, equal-spin binaries in (Rezzolla-et al, Marronetti-et al, Berti-et al, Buonanno-et al 2007),
- Triangles to equal-mass, unequal-spin binaries in (Rezzolla-et al, Berti-et al),
- Squares to unequal-mass, equal-spin binaries in (Berti-et al, Buonanno-et al, Rezzolla-et al).

Misaligned spinning

- Hexagons refer to data from ref. (Campanelli:2006vp)
- Squares to the data obtained in AEI runs
- Circles to data from ref. (Tichy:2007gso)
- Triangles to data from ref. (Herrmann:2007ex)
Binary black holes are a fertile ground for gravitational physics (recoil, spin, waveforms).
- Modelling of final spins and kicks within a few percent precision
- Hybrid methods, combining post-Newtonian and perturbative approaches with numerical results are starting to provide a picture of the full inspiral-merger process.

Techniques for numerical relativity are now rather advanced. There are still systematic problems to be tackled:
- Efficiency.
- Improving initial data construction.
- Understanding limitations of wave extraction at a finite radius.

Room for improvement by including $J_{rad}$ in the orbital plane
Interesting to see what we can discover about extremal spin regions