Constraint Preserving Boundary Treatment for the Einstein Equations in 2nd Order Form

Jennifer Seiler

Collaborators:

B Szilagyi, L Rezzolla

Max-Planck-Institute for Gravitational Physics Potsdam, Germany

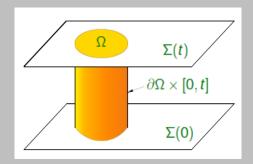
XXX Encuentros Relativistas Españoles Puerto de la Cruz, Tenerife 19th September 2007



IBVP

The Initial Boundary Value Problem

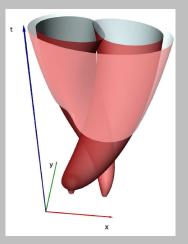
- To simulate spacetimes numerically on a finite grid we truncate the computational domain by introducing an artificial outer boundary.
- The boundary conditions should:
 - be compatible with the constraints
 - reduce reflections
 - yield a well-posed initial-boundary value problem.





The AEIHarmonic Code

- Generalized harmonic system
- 2nd differential order in space
- Constraint damping
- 4th order finite differencing
- Moving lego-excision
- Mesh refinement (with Carpet)



Inspiral and Merger with Harmonic Coordinates. A smooth crossing of the horizons can clearly be seen.

Description Features

"Generalised" Harmonic Coordinates

Coordinates:

- GH coordinates, x^{μ} , satisfy the condition $\Box x^{\mu} = \Gamma^{\mu} = F^{\mu}$.
- $F^{\mu}(g^{\alpha\beta}, x^{\rho})$ as a source function chosen to fine tune gauge to address the requirements of specific simulations.
- Provides solutions of the EEs provided that the constraints:

$$C^{\mu} \equiv \Gamma^{\mu} - \widehat{\Gamma}^{\mu} = rac{1}{\sqrt{-g}}rac{\partial}{\partial x^{\kappa}} \left(\sqrt{-g}g^{\lambda\kappa}
ight) - \widehat{\Gamma}^{\mu} = 0$$

and their time derivatives are initially satisfied.

Evolution Variables:

- We define the evolution variables $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$ and $Q^{\mu\nu} \equiv n^{\rho}\partial_{\rho}\tilde{g}^{\alpha\beta}$, where n^{ρ} is timelike.
- This simplifies the constraint equations to

$$C^{\mu}\equiv -rac{1}{\sqrt{-g}}\partial_{lpha} ilde{g}^{lpha\mu}-\widehat{\Gamma}^{\mu}$$

and gives us a first order in time evolution system.



Features of Generalized Harmonic Coordinates

- System of equations is manifestly symmetric hyperbolic (given reasonable metric conditions).
- Simplifies the evolution equations:
 - When the gradient of this condition is substituted for terms in Einstein equations, the PP of each metric element reduces to a simple wave equation:

$$g^{\gamma\delta}g_{\alpha\beta,\gamma\delta}+\ldots=0$$

• Constraints have the same form.

- The constraint equations may be incorporated into the generalized harmonic coordinate conditions.
- Gauge source terms for Harmonic coordinates allow free choice of gauge for Einstein equations.



Summation by Parts Boundaries

- The SBP method allows us to derive difference operators and boundary condition which control the energy growth of the system and thus provide a mathematically and numerically well-posed system.
- A discrete difference operator is said to satisfy SBP for a scalar product $E=\langle u,v
 angle$ if the property

$$\langle u, Dv \rangle + \langle v, Du \rangle = (u \cdot v) \mid_a^b$$

holds for all functions u, v in [a, b].

• One can construct a 3D SBP operator by applying the 1D operator to each direction. The resulting operator also satisfies SBP with respect to a diagonal scalar product

$$(u, v)_{\Sigma} = h_x h_y h_z \sum_{ijk} \sigma_{ijk} u_{ijk} \cdot v_{ijk},$$

 Using SBP difference operators we can formulate an energy estimate for our evolution system...

Summation by Parts Boundaries

- The SBP method allows us to derive difference operators and boundary condition which control the energy growth of the system and thus provide a mathematically and numerically well-posed system.
- $\circ\,$ A discrete difference operator is said to satisfy SBP for a scalar product $E=\langle u,v\rangle$ if the property

$$\langle u, Dv \rangle + \langle v, Du \rangle = (u \cdot v) \mid_a^b$$

holds for all functions u, v in [a, b].

• One can construct a 3D SBP operator by applying the 1D operator to each direction. The resulting operator also satisfies SBP with respect to a diagonal scalar product

$$(u,v)_{\Sigma} = h_x h_y h_z \sum_{ijk} \sigma_{ijk} u_{ijk} \cdot v_{ijk},$$

• Using SBP difference operators we can formulate an energy estimate for our evolution system...



Summation by Parts Boundaries

- The SBP method allows us to derive difference operators and boundary condition which control the energy growth of the system and thus provide a mathematically and numerically well-posed system.
- $\circ\,$ A discrete difference operator is said to satisfy SBP for a scalar product $E=\langle u,v\rangle$ if the property

$$\langle u, Dv \rangle + \langle v, Du \rangle = (u \cdot v) \mid_a^b$$

holds for all functions u, v in [a, b].

 One can construct a 3D SBP operator by applying the 1D operator to each direction. The resulting operator also satisfies SBP with respect to a diagonal scalar product

$$(u, v)_{\Sigma} = h_x h_y h_z \sum_{ijk} \sigma_{ijk} u_{ijk} \cdot v_{ijk},$$

• Using SBP difference operators we can formulate an energy estimate for our evolution system...

Summation by Parts Boundaries

- The SBP method allows us to derive difference operators and boundary condition which control the energy growth of the system and thus provide a mathematically and numerically well-posed system.
- $\circ\,$ A discrete difference operator is said to satisfy SBP for a scalar product $E=\langle u,v\rangle$ if the property

$$\langle u, Dv \rangle + \langle v, Du \rangle = (u \cdot v) \mid_a^b$$

holds for all functions u, v in [a, b].

 One can construct a 3D SBP operator by applying the 1D operator to each direction. The resulting operator also satisfies SBP with respect to a diagonal scalar product

$$(u, v)_{\Sigma} = h_x h_y h_z \sum_{ijk} \sigma_{ijk} u_{ijk} \cdot v_{ijk},$$

• Using SBP difference operators we can formulate an energy estimate for our evolution system...



Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = ||u(\cdot, t)||^2$ of your system should satisfy $||u(\cdot, t)||^2 \le K(t) ||u(\cdot, 0)||^2$
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
 - Integrate using the SBP rule
 - Substitute our boundary conditions and apply maximally dissipative condition.
 - · Applying that estimate as a penalty to our original evolution equations
 - We can then choose coefficients for our boundary system which control the energy growth of the whole system.

$$\begin{split} \partial_{t}Q^{\mu\nu} &= -\frac{\gamma^{it}}{\gamma^{tt}}D_{i+}Q^{\mu\nu} - (\gamma^{ij} + \frac{\gamma^{it}\gamma^{jt}}{\gamma^{tt}})H^{-1}(A_{ij} + E_{0} - E_{N})S_{i})\gamma^{\mu} \\ &+ \frac{2\gamma^{ij}}{\gamma^{tt}\beta_{0}}H^{-1}E_{0_{i}}[(1 + \frac{\gamma^{it}}{\gamma^{tt}})D_{i+}\gamma^{\mu\nu} - \frac{Q^{\mu\nu}}{\gamma^{tt}} + \frac{2x}{r^{2}}(\gamma^{\mu\nu} - g_{0})] \\ &+ \frac{2\gamma^{ij}}{\gamma^{tt}\beta_{N}}H^{-1}E_{N_{i}}[(1 - \frac{\gamma^{it}}{\gamma^{tt}})D_{i+}\gamma^{\mu\nu} + \frac{Q^{\mu\nu}}{\gamma^{tt}} + \frac{2x}{r^{2}}(\gamma^{\mu\nu} - g_{N})] \end{split}$$



Constraint Preservation

• Sommerfeld-type outgoing conditions:

$$\left(\partial_t - \partial_x - \frac{1}{r}\right)\left(\gamma^{\mu\nu} - \gamma^{\mu\nu}_0\right) = 0$$

 $\,\circ\,$ For CP Boundaries we set the four $\gamma^{t\mu}$ from the constraints:

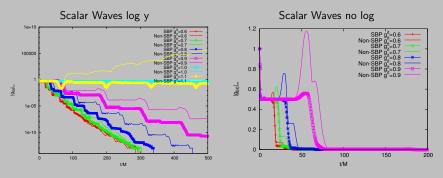
$$C^{\mu} = -\partial_t \gamma^{t\mu} - \partial_i \gamma^{i\mu} - F^{\mu} = 0$$

and we derive a set of outgoing conditions which specify the other 6 metric components:

$$\left(\partial_{x} + \partial_{t} + \frac{1}{r}\right)\left(\gamma^{AB} - \gamma_{0}^{AB}\right) = 0$$
$$\left(\partial_{x} + \partial_{t} + \frac{1}{r}\right)\left(\gamma^{tA} - \gamma^{xA} - \gamma_{0}^{tA} + \gamma_{0}^{xA}\right) = 0$$
$$\left(\partial_{x} + \partial_{t} + \frac{1}{r}\right)\left(\gamma^{tt} - 2\gamma^{xt} + \gamma^{xx} - \gamma_{0}^{tt} + 2\gamma_{0}^{xt} - \gamma_{0}^{xx}\right) = 0$$

see: [2] {Kreiss and Winicour, gr-qc 0602051}

Results for High Shifts



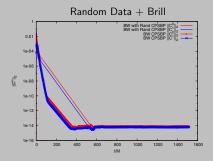
- Tests with Scalarwave testbed
- Stability test for various shifts (0.6 < $\frac{\gamma^{it}}{\gamma^{tt}}$ < 1.1):

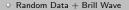
$$u_{tt} = -2rac{\gamma^{it}}{\gamma^{tt}}u_{it} - rac{\gamma^{ij}}{\gamma^{tt}}u_{ij}$$

- Thin = Standard Somerfeld, Thick = SBP
- Reflections for standard BCs clearly visible in right hand plot

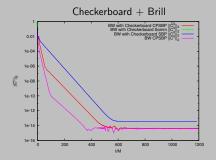


Robust Stability Tests





- \circ Random Kernel Amplitude = 0.1
- Brill Wave Amplitude = 0.5
- dx = 0.2 xmax = 7.1
- Runs stable for in nonlinear regime for Brill Waves.
- Stable for random data
- Standard Sommerfeld type breaks rapidly for this simulation



- Checkerboard Data + Brill Wave
 - for each x(i), y(j), z(k) we add (-1)^{i+j+k}A highest frequency noise possible
 - Checker Kernel $A = \pm 0.2$
 - $_{\odot}\,$ Brill Wave Amplitude = 0.5
 - $\circ \ dx = 0.2 \ xmax = 7.1$
- Standard sommerfeld seen in green (breaks quickly)

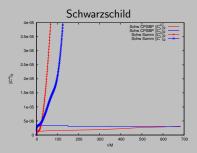


CP SBP Boundaries 2nd Order

Results for Teukolsky/Brill Wave and Schwarschild Runs



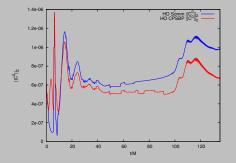
- High Amplitude Teukolsky WavesConstraint Norms for runs with:
 - Constraint Preserving 'SBP' = Red
 - Pure SBP = Magenta
 - Standard sommerfeld-type = Blue

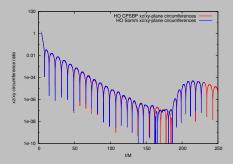


- Schwarzschild run with boundaries too close in (40 M) for sommerfeld-type boundaries
- CPSBP remains stable



Head-on Runs with CPSBP

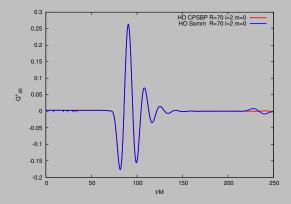




- Headon Collison (mass 0.5, 2.5 M separation)
- L2 Norm of Constraints for CPSBP vs regular boundaries
- Significant improvement in constraint preservation

- Circumference ratios almost identical
- Some boundary effects are visible for the standard BC runs which are not in the CPSBP run

Conclusions



- SBP provides a provably well-posed and demonstrably stable IBVP for Generalized Harmonic evolutions on a Cartesian grid
- Stands up to stability tests
- We have developed a method which allows us to consistently use SBP on a Cartesian grid for corners and edges, and for a 2nd order in space system



Thank You.





Discrete Boundary Treatment for the Shifted Wave Equation in Second Order Form and Related Problems.



Kreiss and Winicour, gr-qc 0602051

Problems Which are Well-Posed in a Generalised Sense With Applications to the Einstein Equations.



[G. Calabrese, J. Pullin, O. Reula, O. Sarbach, and M. Tiglio] Well Posed Constraint-preserving Boundary Conditions For the Linearized Einstein Equations.



Initial Boundary Value Problem for Einstein's Vacuum Field Equation.



Well-posed Initial-boundary Evolution in General Relativity.



Constraints

Constraint Damping

- The constraint equations are the generalized harmonic coordinate conditions: $C^{\mu}\equiv\Gamma^{\mu}-\widehat{\Gamma}^{\mu}=0$
- o constraint adjustment is done by the term

$$A^{\mu
u} = C^{
ho} A^{\mu
u}_{
ho} \left(x^{lpha}, g_{lphaeta}, \partial_{\gamma} g_{lphaeta}
ight)$$

in the evolution equations

$$\partial_{\alpha} \left(g^{lphaeta} \partial_{eta} \tilde{g}^{\mu
u}
ight) + S^{\mu
u} \left(g, \partial g
ight) + \sqrt{-g} \mathcal{A}^{\mu
u}$$

 $+ 2\sqrt{-g} \nabla^{(\mu} F^{\nu)} - \tilde{g}^{\mu
u} \nabla_{lpha} F^{lpha} = 0.$

• Dissipation: $\dot{f} \longrightarrow \dot{f} + \epsilon (\delta^{ij} D_{+i} D_{-i}) w (\delta^{ij} D_{+i} D_{-i}) f$ where w is a weight factor that vanishes at the outer boundary. With $D_{+i} D_{-i}$ from blended SBP stencils.



HarmonicExcision

 $(n^i D_{+i})^3 \dot{f} = 0$ to all guard points, in layers stratified by length of the outward normal pointing vector, from out to in. LegoExcision with excision coefficients $\frac{x^{\mu}}{r}$ extrapolated around a smooth virtual surface for the inner boundary. Radiation outer boundary conditions (i.e. outgoing only).

Past Work

- [*Stewart* 1998] Necessary conditions for well-posedness of linearized Einstein equations with constraint-preserving boundary conditions (Fourier-Laplace analysis)
- [*Friedrich & Nagy* 1999] To-date the only formulation proven to satisfy all the requirements for the *fully nonlinear* (vacuum) Einstein equations (frame formalism)
- [*Kreiss & Winicour* 2006] Well posed and constraint preserving boundary conditions for *linearized* Einstein Equations



Penalty Method

Wł

For the harmonic system the interior is:

$$\partial_t Q^{\mu\nu} = rac{\gamma^{it}}{\gamma^{tt}} D_{i+} Q^{\mu\nu} - (\gamma^{ij} + rac{\gamma^{it} \gamma^{jt}}{\gamma^{tt}}) H^{-1} A_{ij} \gamma^{\mu\nu}$$

With the Boundaries it is:

$$\begin{split} \partial_t Q^{\mu\nu} &= -\frac{\gamma^{it}}{\gamma^{tt}} D_{i+} Q^{\mu\nu} - (\gamma^{ij} + \frac{\gamma^{it}\gamma^{jt}}{\gamma^{tt}}) H^{-1} (A_{ij} + E_0 - E_N) S_i) \gamma^{\mu\nu} \\ &+ \frac{2\gamma^{ij}}{\gamma^{tt}\beta_0} H^{-1} E_{0_i} [(1 + \frac{\gamma^{it}}{\gamma^{tt}}) D_{i+} \gamma^{\mu\nu} - \frac{Q^{\mu\nu}}{\gamma^{tt}} + \frac{2x}{r^2} (\gamma^{\mu\nu} - g_0)] \\ &+ \frac{2\gamma^{ij}}{\gamma^{tt}\beta_N} H^{-1} E_{N_i} [(1 - \frac{\gamma^{it}}{\gamma^{tt}}) D_{i+} \gamma^{\mu\nu} + \frac{Q^{\mu\nu}}{\gamma^{tt}} + \frac{2x}{r^2} (\gamma^{\mu\nu} - g_N)] \\ \text{here } \gamma_{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} \text{ and } Q^{\mu\nu} = g^{t\alpha} \partial_\alpha \gamma^{\mu\nu}. \end{split}$$



"Generalised" Harmonic Coordinates

- GH coordinates, x^{μ} , satisfy the condition $\Box x^{\mu} = \Gamma^{\mu} = F^{\mu}$. With the d'Alembertian, $\Box \phi \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\lambda}} \left(\sqrt{-g} g^{\lambda \kappa} \frac{\partial \phi}{\partial x^{\kappa}} \right)$
- GH coordinates coupled to the Einstein Equations gives:

• Gauge freedom from the ability to pick the four $\widehat{\mathsf{\Gamma}}^{\mu}(g^{lphaeta},x^{
ho}).$

AEIHarmonic Evolution

- We define the evolution variables $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$ and $Q^{\mu\nu} \equiv n^{\rho}\partial_{\rho}\tilde{g}^{\alpha\beta}$, where n^{ρ} is timelike.
- o This simplifies the constraint equations to

$$C^{\mu}\equiv -rac{1}{\sqrt{-g}}\partial_{lpha} ilde{g}^{lpha\mu}-\widehat{\Gamma}^{\mu}$$

• The AEIHarmonic code implements the first order in time system:

$$\partial_t \tilde{g}^{\mu\nu} = -\frac{g^{it}}{g^{tt}} \partial_i \tilde{g}^{\mu\nu} + \frac{1}{g^{tt}} Q^{\mu\nu}$$
$$\partial_t Q^{\mu\nu} = -\partial_i \left(\left(g^{ij} - \frac{g^{it}g^{jt}}{g^{tt}} \right) \partial_j \tilde{g}^{\mu\nu} \right) - \partial_i \left(\frac{g^{it}}{g^{tt}} Q^{\mu\nu} \right) + \tilde{S}^{\mu\nu}$$

