# Constraint Preserving Boundary Treatment for the Einstein Equations in 2nd Order Form

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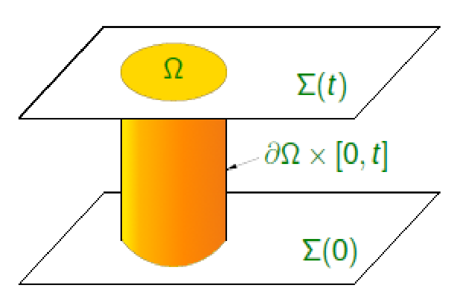
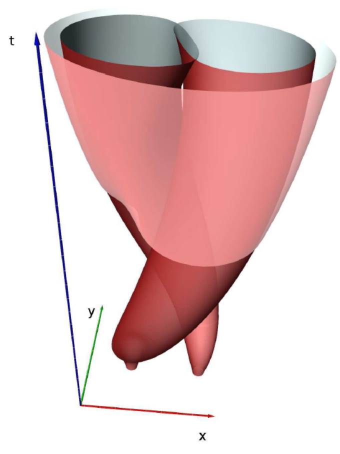
# The Initial Boundary Value Problem

- To simulate spacetimes numerically on a finite grid we truncate the computational domain by introducing an artificial outer boundary.
- The boundary conditions should:
  - be compatible with the constraints
  - reduce reflections
  - yield a well-posed initial-boundary value problem.



# The AEIHarmonic Code

- Generalized harmonic system
- 2nd differential order in space
- Constraint damping
- 4th order finite differencing
- Moving lego-excision
- Mesh refinement (with Carpet)




Inspiral and Merger with Harmonic Coordinates. A smooth crossing of the horizons can clearly be seen.

# "Generalised" Harmonic Coordinates

Coordinates:

- GH coordinates, $x^\mu$, satisfy the condition $\Box x^\mu = \Gamma^\mu = F^\mu$.
- $F^\mu(g^{\alpha\beta}, x^\rho)$ as a source function chosen to fine tune gauge to address the requirements of specific simulations.
- Provides solutions of the EEs provided that the constraints:

$$C^\mu \equiv \Gamma^\mu - \widehat{\Gamma}^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\kappa} \left( \sqrt{-g} g^{\lambda\kappa} \right) - \widehat{\Gamma}^\mu = 0$$

and their time derivatives are initially satisfied.

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Evolution Variables:

- We define the evolution variables $\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$ and $Q^{\mu\nu} \equiv n^\rho \partial_\rho \tilde{g}^{\alpha\beta}$, where $n^\rho$ is timelike.
- This simplifies the constraint equations to

$$C^\mu \equiv -\frac{1}{\sqrt{-g}} \partial_\alpha \tilde{g}^{\alpha\mu} - \widehat{\Gamma}^\mu$$

and gives us a first order in time evolution system.

# Features of Generalized Harmonic Coordinates

- System of equations is manifestly symmetric hyperbolic (given reasonable metric conditions).
- Simplifies the evolution equations:
  - When the gradient of this condition is substituted for terms in Einstein equations, the PP of each metric element reduces to a simple wave equation:

$$g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + \ldots = 0$$

  - Constraints have the same form.
- The constraint equations may be incorporated into the generalized harmonic coordinate conditions.
- Gauge source terms for Harmonic coordinates allow free choice of gauge for Einstein equations.

# Summation by Parts Boundaries

- The SBP method allows us to derive difference operators and boundary condition which control the energy growth of the system and thus provide a mathematically and numerically well-posed system.
- A discrete difference operator is said to satisfy SBP for a scalar product $E = \langle u, v \rangle$ if the property

$$\langle u, Dv \rangle + \langle v, Du \rangle = (u \cdot v)\,|_a^b$$

holds for all functions $u, v$ in $[a, b]$.

- One can construct a 3D SBP operator by applying the 1D operator to each direction. The resulting operator also satisfies SBP with respect to a diagonal scalar product

$$(u, v)_\Sigma = h_x h_y h_z \sum_{ijk} \sigma_{ijk} u_{ijk} \cdot v_{ijk},$$

- Using SBP difference operators we can formulate an energy estimate for our evolution system...

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# Well-Posed Boundaries

- For well-posedness, the energy estimate $\xi^{(n)} = \| u(\cdot, t) \|^2$ of your system should satisfy $\| u(\cdot, t) \|^2 \leq K(t) \| u(\cdot, 0) \|^2$
- We use the SBP rule to derive an estimate for the time derivative of the energy of the system.
  - Integrate using the SBP rule
  - Substitute our boundary conditions and apply maximally dissipative condition.
  - Applying that estimate as a penalty to our original evolution equations
  - We can then choose coefficients for our boundary system which control the energy growth of the whole system.

$$\partial_t Q^{\mu\nu} = -\frac{\gamma^{it}}{\gamma^{tt}} D_{i+} Q^{\mu\nu} - (\gamma^{ij} + \frac{\gamma^{it}\gamma^{jt}}{\gamma^{tt}}) H^{-1}(A_{ij} + E_0 - E_N) S_i)\gamma^{\mu\nu}$$

$$+\frac{2\gamma^{ij}}{\gamma^{tt}\beta_0} H^{-1} E_{0_i}[(1 + \frac{\gamma^{it}}{\gamma^{tt}}) D_{i+}\gamma^{\mu\nu} - \frac{Q^{\mu\nu}}{\gamma^{tt}} + \frac{2x}{r^2}(\gamma^{\mu\nu} - g_0)]$$

$$+\frac{2\gamma^{ij}}{\gamma^{tt}\beta_N} H^{-1} E_{N_i}[(1 - \frac{\gamma^{it}}{\gamma^{tt}}) D_{i+}\gamma^{\mu\nu} + \frac{Q^{\mu\nu}}{\gamma^{tt}} + \frac{2x}{r^2}(\gamma^{\mu\nu} - g_N)]$$

# Constraint Preservation

- Sommerfeld-type outgoing conditions:

$$\left(\partial_t - \partial_x - \frac{1}{r}\right)(\gamma^{\mu\nu} - \gamma_0^{\mu\nu}) = 0$$

- For CP Boundaries we set the four $\gamma^{t\mu}$ from the constraints:

$$C^\mu = -\partial_t \gamma^{t\mu} - \partial_i \gamma^{i\mu} - F^\mu = 0$$

and we derive a set of outgoing conditions which specify the other 6 metric components:

$$\left(\partial_x + \partial_t + \frac{1}{r}\right)(\gamma^{AB} - \gamma_0^{AB}) = 0$$

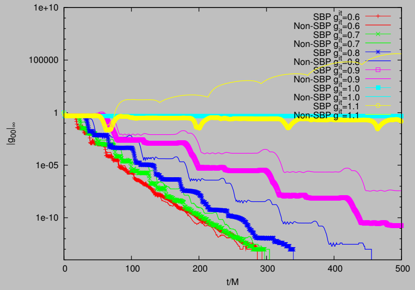
$$\left(\partial_x + \partial_t + \frac{1}{r}\right)(\gamma^{tA} - \gamma^{xA} - \gamma_0^{tA} + \gamma_0^{xA}) = 0$$

$$\left(\partial_x + \partial_t + \frac{1}{r}\right)(\gamma^{tt} - 2\gamma^{xt} + \gamma^{xx} - \gamma_0^{tt} + 2\gamma_0^{xt} - \gamma_0^{xx}) = 0$$

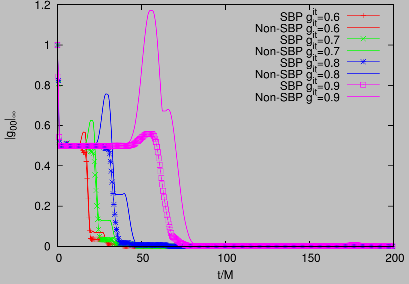
see: [2] {Kreiss and Winicour, gr-qc 0602051}

# Results for High Shifts

Scalar Waves log y




Scalar Waves no log



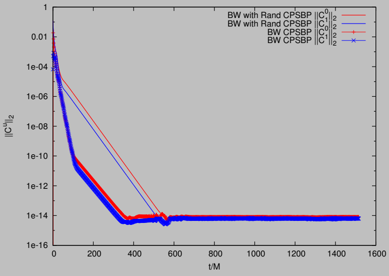

- Tests with Scalarwave testbed
- Stability test for various shifts ($0.6 < \frac{\gamma^{it}}{\gamma^{tt}} < 1.1$):

$$u_{tt} = -2\frac{\gamma^{it}}{\gamma^{tt}}u_{it} - \frac{\gamma^{ij}}{\gamma^{tt}}u_{ij}$$

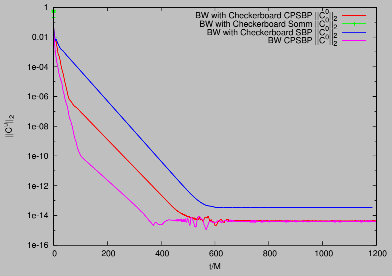
- Thin = Standard Somerfeld, Thick = SBP
- Reflections for standard BCs clearly visible in right hand plot

# Robust Stability Tests

## Random Data + Brill




## Checkerboard + Brill

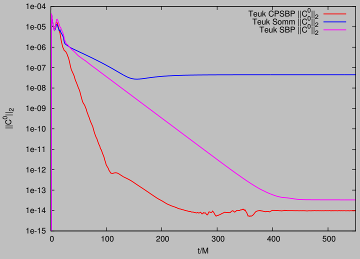



- Random Data + Brill Wave
  - Random Kernel Amplitude = 0.1
  - Brill Wave Amplitude = 0.5
  - dx = 0.2 xmax = 7.1
- Runs stable for in nonlinear regime for Brill Waves.
- Stable for random data
- Standard Sommerfeld type breaks rapidly for this simulation

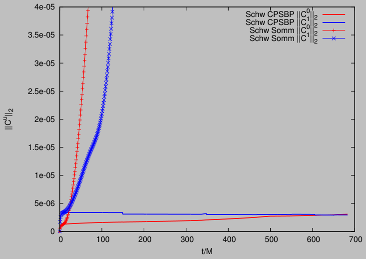
- Checkerboard Data + Brill Wave
  - for each $x(i), y(j), z(k)$ we add $(-1)^{i+j+k}A$ highest frequency noise possible
  - Checker Kernel $A = \pm 0.2$
  - Brill Wave Amplitude = 0.5
  - dx = 0.2 xmax = 7.1
- Standard sommerfeld seen in green (breaks quickly)

# Results for Teukolsky/Brill Wave and Schwarzschild Runs

Teukolsky



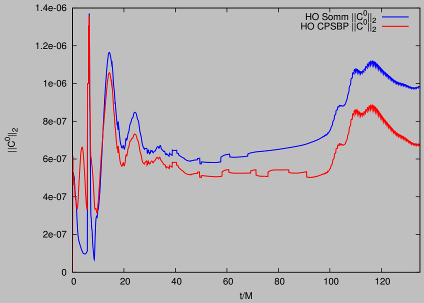

Schwarzschild

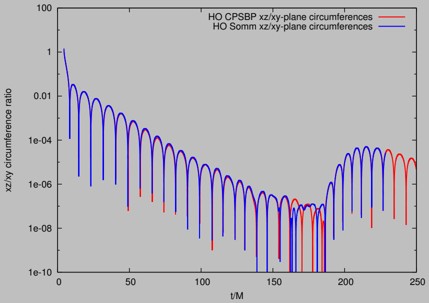



- High Amplitude Teukolsky Waves
- Constraint Norms for runs with:
  - Constraint Preserving 'SBP' = Red
  - Pure SBP = Magenta
  - Standard sommerfeld-type = Blue

- Schwarzschild run with boundaries too close in (40 M) for sommerfeld-type boundaries
- CPSBP remains stable

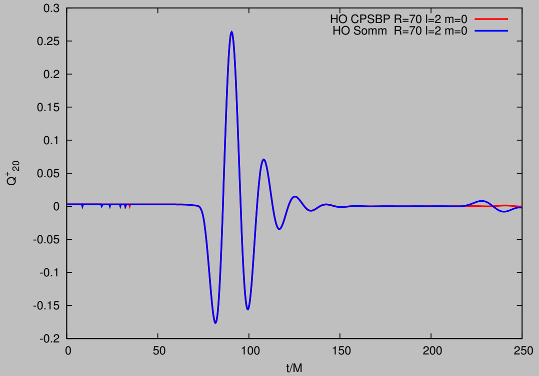
# Head-on Runs with CPSBP




- Headon Collison (mass 0.5, 2.5 M separation)
- L2 Norm of Constraints for CPSBP vs regular boundaries
- Significant improvement in constraint preservation
- Circumference ratios almost identical
- Some boundary effects are visible for the standard BC runs which are not in the CPSBP run

# Conclusions




- SBP provides a provably well-posed and demonstrably stable IBVP for Generalized Harmonic evolutions on a Cartesian grid
- Stands up to stability tests
- We have developed a method which allows us to consistently use SBP on a Cartesian grid for corners and edges, and for a 2nd order in space system

Thank You.

- [G. Calabrese and C. Gundlach, gr-qc 0509119]
  Discrete Boundary Treatment for the Shifted Wave Equation in Second Order Form and Related Problems.
  *General Relativity and Quantum Cosmology*, 0509119, 31 July 2006.

- [Kreiss and Winicour, gr-qc 0602051]
  Problems Which are Well-Posed in a Generalised Sense With Applications to the Einstein Equations.
  *General Relativity and Quantum Cosmology*, 0602051, 6 June 2006.

- [G. Calabrese, J. Pullin, O. Reula, O. Sarbach, and M. Tiglio]
  Well Posed Constraint-preserving Boundary Conditions For the Linearized Einstein Equations.
  *Comm. Math. Phys.*, 240:377395, 5 Sept. 2002.

- [H. Friedrich and G. Nagy, Comm. Math. Phys. 201]
  Initial Boundary Value Problem for Einstein's Vacuum Field Equation.
  *Comm. Math. Phys.*, 201:619-655, 15 Sept. 1999.

- [B. Szilagyi and J. Winicour, PRD 68:041501]
  Well-posed Initial-boundary Evolution in General Relativity.
  *Phys. Rev. D*, 68:041501(1)041501(5), 2003.

# Constraint Damping

- The constraint equations are the generalized harmonic coordinate conditions: $C^\mu \equiv \Gamma^\mu - \widehat{\Gamma}^\mu = 0$
- constraint adjustment is done by the term

$$A^{\mu\nu} = C^\rho A^{\mu\nu}_\rho \left(x^\alpha, g_{\alpha\beta}, \partial_\gamma g_{\alpha\beta}\right)$$

in the evolution equations

$$\partial_\alpha \left(g^{\alpha\beta}\partial_\beta \tilde{g}^{\mu\nu}\right) + S^{\mu\nu}\left(g, \partial g\right) + \sqrt{-g}A^{\mu\nu}$$

$$+2\sqrt{-g}\nabla^{(\mu}F^{\nu)} - \tilde{g}^{\mu\nu}\nabla_\alpha F^\alpha = 0.$$

- Dissipation: $\dot{f} \longrightarrow \dot{f} + \epsilon(\delta^{ij}D_{+i}D_{-i})w(\delta^{ij}D_{+i}D_{-i})f$ where $w$ is a weight factor that vanishes at the outer boundary. With $D_{+i}D_{-i}$ from blended SBP stencils.

# HarmonicExcision

$\left(n^i D_{+i}\right)^3 \dot{f} = 0$ to all guard points, in layers stratified by length of the outward normal pointing vector, from out to in.

LegoExcision with excision coefficients $\dfrac{x^\mu}{r}$ extrapolated around a smooth virtual surface for the inner boundary.

Radiation outer boundary conditions (i.e. outgoing only).

# Past Work

- [*Stewart* 1998] Necessary conditions for well-posedness of linearized Einstein equations with constraint-preserving boundary conditions (Fourier-Laplace analysis)
- [*Friedrich & Nagy* 1999] To-date the only formulation proven to satisfy all the requirements for the *fully nonlinear* (vacuum) Einstein equations (frame formalism)
- [*Kreiss & Winicour* 2006] Well posed and constraint preserving boundary conditions for *linearized* Einstein Equations

# Penalty Method

For the harmonic system the interior is:

$$\partial_t Q^{\mu\nu} = \frac{\gamma^{it}}{\gamma^{tt}} D_{i+} Q^{\mu\nu} - (\gamma^{ij} + \frac{\gamma^{it}\gamma^{jt}}{\gamma^{tt}}) H^{-1} A_{ij} \gamma^{\mu\nu}$$

With the Boundaries it is:

$$\partial_t Q^{\mu\nu} = -\frac{\gamma^{it}}{\gamma^{tt}} D_{i+} Q^{\mu\nu} - (\gamma^{ij} + \frac{\gamma^{it}\gamma^{jt}}{\gamma^{tt}}) H^{-1} (A_{ij} + E_0 - E_N) S_i) \gamma^{\mu\nu}$$

$$+ \frac{2\gamma^{ij}}{\gamma^{tt}\beta_0} H^{-1} E_{0_i} [(1 + \frac{\gamma^{it}}{\gamma^{tt}}) D_{i+} \gamma^{\mu\nu} - \frac{Q^{\mu\nu}}{\gamma^{tt}} + \frac{2x}{r^2}(\gamma^{\mu\nu} - g_0)]$$
$$+ \frac{2\gamma^{ij}}{\gamma^{tt}\beta_N} H^{-1} E_{N_i} [(1 - \frac{\gamma^{it}}{\gamma^{tt}}) D_{i+} \gamma^{\mu\nu} + \frac{Q^{\mu\nu}}{\gamma^{tt}} + \frac{2x}{r^2}(\gamma^{\mu\nu} - g_N)]$$

Where $\gamma_{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$ and $Q^{\mu\nu} = g^{t\alpha}\partial_\alpha \gamma^{\mu\nu}$.

# "Generalised" Harmonic Coordinates

- GH coordinates, $x^\mu$, satisfy the condition $\Box x^\mu = \Gamma^\mu = F^\mu$.
  With the d'Alembertian, $\Box\phi \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\lambda\kappa}\frac{\partial\phi}{\partial x^\kappa}\right)$
- GH coordinates coupled to the Einstein Equations gives:

$$G_{\mu\nu} = (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 8\pi T_{\mu\nu} \Longrightarrow$$

$$\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + g_{\alpha(\mu}\partial_{\nu)}\Gamma^\alpha + F_{\mu\nu}(g, \partial_g) = 0$$

- Gauge freedom from the ability to pick the four $\widehat{\Gamma}^\mu(g^{\alpha\beta}, x^\rho)$.

# AEIHarmonic Evolution

- We define the evolution variables $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu}$ and $Q^{\mu\nu} \equiv n^\rho \partial_\rho \tilde{g}^{\alpha\beta}$, where $n^\rho$ is timelike.
- This simplifies the constraint equations to

$$C^\mu \equiv -\frac{1}{\sqrt{-g}} \partial_\alpha \tilde{g}^{\alpha\mu} - \widehat{\Gamma}^\mu$$

- The AEIHarmonic code implements the first order in time system:

$$\partial_t \tilde{g}^{\mu\nu} = -\frac{g^{it}}{g^{tt}} \partial_i \tilde{g}^{\mu\nu} + \frac{1}{g^{tt}} Q^{\mu\nu}$$

$$\partial_t Q^{\mu\nu} = -\partial_i \left( \left( g^{ij} - \frac{g^{it} g^{jt}}{g^{tt}} \right) \partial_j \tilde{g}^{\mu\nu} \right) - \partial_i \left( \frac{g^{it}}{g^{tt}} Q^{\mu\nu} \right) + \tilde{S}^{\mu\nu}$$